Stability of a planar flame front in the slow-combustion regime

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The hydrodynamic stability of a planar flame (deflagration) is determined by solving the complete system of equations, including thermal conduction and energy release due to chemical reactions, for the case in which the Lewis number is equal to unity. In the asymptotic limit of large-wavelength perturbations, the developed theory provides a rigorous justification of the Darrieus-Landau assumption that the flame-front velocity is constant, which is the necessary supplementary condition in the model of discontinuous flame front. The analytical solution for the suppression of the flame-front instability is obtained for an arbitrary activation energy. It is shown that the obtained solution does not depend on the specific form of the energy release. The perturbation growth rate is also found numerically by solving the eigenvalue problem.

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I. INTRODUCTION

The stability analysis for many problems in hydrodynamics and plasma physics requires a solution of the complicated system of equations including the energy release due to reactions, dissipations, thermal conductivity equations, etc. The problem can be essentially simplified if the characteristic scales of some processes for the unperturbed flow are small in comparison with the characteristic dimensions of the problem. In this case, one may simplify the problem by regarding the thin zone at the position where the large gradients of the variables are located (zone of the energy release, chemical reactions, temperature gradient, etc.) as a surface of discontinuity of zero thickness. Thus formulated, the problem of the stability can in a certain sense be separated from the kinetic and/or transport problems and can be regarded as the purely hydrodynamic stability problem of a surface of discontinuity of zero thickness. A number of boundary conditions (jump conditions at the discontinuity surface) must be satisfied if one introduces a discontinuity instead of the transition layer. As is known, the necessary condition, that the solution for the flow with a discontinuity exists, is provided by the so-called "condition of evolutionary" of the flow [1–3]. The point is that any initial small perturbation is defined by some number of independent parameters for the given flow. Its development is governed by a set of the linearized boundary conditions at the discontinuity surface. The evolutionary condition is satisfied if the number of conditions at the discontinuity is one greater than the total number of simple waves that can propagate out of the discontinuity surface. In general, the boundary conditions at the discontinuity surface can be divided into the basic boundary conditions which follow from the conservation laws being the integrals of the hydrodynamic equations, and the additional boundary condition which does not follow from the hydrodynamic equations [1–3]. This additional boundary condition can be obtained by solving the complete problem, including the processes inside of the discontinuity zone. In some sense the supplementary boundary condition guarantees the existence of the stationary solution for the flow with the discontinuity surface. For example, in the problem of flame propagation, regarding the flame front as a discontinuity surface, this additional condition is the given velocity of the flame front. This velocity can be found in turn as the eigenvalue of the *complete* combustion problem, including the mechanism for flame propagation, i.e., thermal conduction and energy release in the chemical reaction [4].

It is obvious that the stability problem for a flow with a discontinuity surface also requires an additional condition on the perturbed hydrodynamic variables if the additional condition is required for the unperturbed steady flow according to the condition of evolutionary of the flow. Sometimes this additional condition can be assumed *phenomenologically* from physical considerations. A well-known example of such a problem is the stability problem of the planar flame front [5,6]. Another problem, where similar difficulties arise due to the lack of the boundary condition, is, for example, the stability analysis of ablatively accelerated laser targets in the approximation where one model the ablation front by a discontinuity. Thus formulated, this problem leads to a similar difficulty: the number of boundary conditions at the ablation surface is insufficient to determine the solution. Many papers have been published [7–10] in which attempts have been made to resolve the problem and describe the suppression of the Rayleigh-Taylor instability due to the mass flow across the ablation front. There are many other problems of similar nature: ionizing shock wave in a magnetic field [2], phase transition wave, etc. In this paper a rigorous solution is obtained for the problem of the hydrodynamic stability of a flame front by integrating the complete set of equations including the chemical kinetic of the reaction and the thermal conduc-

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tion of the gas which are responsible for the flame propagation. The obtained solution demonstrates the rigorous approach to problems of a similar nature.

Many papers have been published (see, for example, Refs. [5,6,11–18]) starting from the first papers by Darrieus [5] and Landau [6] where authors made attempts to solve the problem of the flame stability. However, all authors used models which included an unjustified assumption. Since in the majority of situations of practical interest the thickness of the combustion zone $\Delta \approx \chi_1/u_1$ is small in comparison with the characteristic dimensions of the problem (χ_1 is the thermal diffusivity of the gas and u_1 is the flame speed relative to the original gas), the purely hydrodynamic problem can in a certain sense be separated from the chemical kinetic problem, and the flame can be regarded as a surface of discontinuity of zero thickness (a flame front), separating the combustion products and the unburnt gas mixture. Thus formulated the stability problem of the flame front was solved by Darrieus and Landau who used an additional assumption that the normal propagation velocity of the flame relative to the unburnt gas remains unchanged in the presence of perturbations. Their solution for the growth rate of perturbations σ as a function of the wave number $k = 2\pi/\lambda$ is

$$\sigma = k u_1 \frac{\Theta_2}{1+\Theta_2} \left[\left[1+\Theta_2 - \frac{1}{\Theta_2} \right]^{1/2} - 1 \right], \tag{1}$$

where $\Theta_2 = T_2/T_1 = \rho_1/\rho_2$ is the thermal expansion the gas undergoes in the flame, i.e., the ratio of the temperatures (densities) in the gas ahead of the flame front (subscript 1) and behind the front (subscript 2). This result means that a thin planar flame front is absolutely unstable to an arbitrary perturbation which bends the flame front. In fact, this result is correct only for long-wavelength perturbations, in the limit $\lambda \gg \Delta$ and the strong decrease in the growth rate of the instabilities must occur at least around $\lambda \approx \Delta$. On the other hand, experiments reveal that the instability growth rate is substantially less than that of Eq. (1) even for fairly long wavelengths $\lambda \sim 10^2 \Delta \gg \Delta$. Since result (1) contradicts strongly the experimental facts, many papers have been published in which attempts have been made to resolve this contradiction. To discuss them in detail would take us far outside the scope of the present work. We restrict ourself to mentioning the paper [12] where the phenomenological assumption has been done so that the flame velocity relative to the unperturbed gas depends on the front curvature. Such a dependence of the flame velocity was associated with the transverse diffusion and the thermal conductivity and yielded the phenomenological correction terms for Eq. (1). A more consistent approach to the problem was developed in Refs. [14–18] where authors considered the problem by solving equations including the thermal conductivity equation and diffusion. However, all authors considered the case of the infinite activation energy $E/T_2 \gg 1$ that, in turn, allowed them to regard the zone of the chemical reaction as a surface of discontinuity of zero thickness (inner discontinuity surface). Similar to the pure hydrodynamic model, one additional boundary condition on the inner discontinuity surface has been introduced *a priori* in these models. First, the problem of the boundary conditions for the inner discontinuity surface (front of the chemical reaction) was treated in [19] (Barenblatt and Zel'dovich). These authors assumed that the concentration $\tilde{a}$ and the temperature perturbations $\tilde{T}$ are continuous and the reaction rate depends on the temperature as $\exp(-E/T)$. The last assumption means the following estimate: $\Delta(d\tilde{T}/dz) \sim (E/T_2)\tilde{T}$. Because the thickness of the reaction zone is of order $(T_2/E)\Delta$, the assumption that $\tilde{T}$ is continuous is a contradiction. Attempts to avoid this contradiction were made in [20,21] where instead of the real dependence for the rate of chemical reaction authors used an artificial assumption about the reaction rate, they used the expression $A \exp(H/2)$, where H is the enthalpy, instead of the realistic strong dependence. Their solution implied the assumption that the perturbations do not change coefficient A and that it is the same as it is for the steady flow. However, in general, the jump conditions for the perturbed values do not follow from the jump conditions for a steady flow. Furthermore, it can be shown that the condition that the perturbed temperature $\tilde{T}$ and concentration $\tilde{a}$ are continuous (used, for example, in [18]) contradicts the linearized equations of thermal conduction and diffusion inside of the reaction zone. The exception is the particular case when Le$=1$, $\tilde{T}=0$, and $\tilde{a}=0$ in the inner discontinuity surface. In this particular case the solution obtained in [15–18] for the infinite activation energy is in agreement with the rigorous solution obtained in this paper.

In this paper a complete rigorous solution is obtained for the problem of the flame stability, including the chemical kinetic of the reaction and the thermal conduction of the gas. It is shown that in the limit of long-wavelength perturbations ($k \to 0$) the condition, that the flame velocity relative to the unperturbed gas remains unchanged, follows from the complete system of linearized equations. A solution of the stability problem is obtained for the case Le$=1$ and for an arbitrary activation energy in explicit analytical form; the numerical solution of the problem is presented. The obtained analytical solution of the problem does not depend explicitly on the particular function of the energy release and can be easily generalized for the wide class of similar problems. For the infinite activation energy the obtained solution is in agreement with the results of papers [15–18] for the case Le $=1$.

In Sec. II the starting equations are presented and the stationary solution is determined. In Sec. III the eigenvalue problem for small perturbations is formulated. The equations and boundary conditions for the perturbations are derived. In Sec. IV the solution is obtained in the limit $k \to 0$, the supplementary condition that the flame velocity remains unchanged, and the asymptotic expression (1) for the growth rate are rigorously justified. In Sec. V the reduction of the growth rate due to thermal conduction is considered and an expression for the "cutoff" wavelength at which the growth rate vanishes is obtained. The numerical solution of the problem is presented in Sec. VI. We conclude in Sec. VII by discussing the analytical and numerical solutions of the problem.

II. STATIONARY SOLUTION OF THE PROBLEM OF THE FLAME PROPAGATION IN THE SLOW-COMBUSTION REGIME

We consider the problem of a slow-combustion wave, taking into account the structure of the transition region (the combustion zone) of the flame front in the case most often encountered in practice, in which the Lewis number satisfies $\mathrm{Le}=\rho c_p D/\kappa=1$, where D and κ are the diffusivity and thermal conductivity of the gas, c_p is the specific heat at constant pressure, and ρ is the gas density. Since the velocity of the flame is small in comparison with the sound speed, the flow can be treated as isobaric. It can be shown [4] that the equality of the diffusivity and thermal conductivity implies that for the steady flow the temperature and the concentration of the fuel mixture have similar profiles, so that the thermal conduction and diffusion equations reduce to a single equation. This assumption is not essential from the physical point of view; it is equivalent to the assumption that the fuel mixture consists of gases with similar molecular weights and that the actual combustion process can be accurately described by a single simple reaction.

Under these conditions the system of equations describing subsonic ($M=u/\sqrt{\gamma P/\rho}\ll 1$) flame propagation takes the form

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0\ , \qquad (2)$$

$$\rho\frac{\partial\mathbf{u}}{\partial t}+(\rho\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla P=0\ , \qquad (3)$$

$$\rho c_p\left[\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T\right]=\nabla\cdot(\kappa\nabla T)+QW(a,T)\ , \qquad (4)$$

where $P=(\gamma-1)c_v\rho T$ is the gas pressure; c_p and c_v are the specific heats at constant pressure and volume; $\gamma=c_p/c_v$ is the ratio of specific heats; Q is the heat and $W(a,T)$ is the rate of the chemical reaction; and a is the concentration of the reacting mixture.

Assuming the Arrhenius law for a first-order chemical reaction, taking into account the similarity of the temperature and the concentration profiles for the steady flow, and assuming that this reaction goes to completion within the combustion zone, i.e., $T_2=T_1+a_1Q/c_p$, we have

$$QW=\frac{\rho c_p}{\tau}(T_2-T)\exp(-E/T)\ , \qquad (5)$$

where E is the activation energy of the chemical reaction and τ is the time dimensional constant of the reaction.

Let us consider one-dimensional planar flow with the z axis parallel to the direction in which the flame is propagating. In the comoving system of coordinates at $z=-\infty$ a one-dimensional stream of unburnt gas of density ρ_1 flows toward the combustion zone with the velocity u_{z1}, while at $z=+\infty$ the combustion products flow away with a density ρ_2 and velocity u_{z2}. For the assumed steady subsonic flow with Mach number $M\ll 1$ the following obvious constants of motion hold:

$$\rho u_z=\mathrm{const}\ , \qquad (6)$$

$$P=\mathrm{const}\ . \qquad (7)$$

Let us go over to dimensionless variables, scaling all quantities with their values at $z=-\infty$. When we include Eqs. (6) and (7), setting $\Theta=T/T_1$, we obtain

$$\frac{\rho_1}{\rho}=\frac{u_z}{u_{z1}}=\frac{T}{T_1}=\Theta\ . \qquad (8)$$

The temperature plays the role of an eigenfunction of the eigenvalue problem determined by Eq. (4) together with Eqs. (6) and (7) and the corresponding boundary conditions. The eigenvalue determined by the boundary conditions is the flame speed. In solving Eq. (4) it is convenient to introduce the dimensionless activation energy $\mathcal{E}=E/T_1$ and the dimensionless position $\xi=z/\Delta$, where $\Delta=\kappa/(\rho c_p u_{z1})$ is the thickness of the combustion zone. Then we can introduce the eigenvalue $\Lambda=\Delta/(u_{z1}\tau)$. In terms of these variables Eq. (4) takes the form

$$\frac{d^2\Theta}{d\xi^2}-\frac{d\Theta}{d\xi}-\Lambda(1-\Theta_2/\Theta)\exp(-\mathcal{E}/\Theta)=0\ , \qquad (9)$$

with boundary conditions

$$\Theta=\begin{cases}1 & \text{for } \xi\to-\infty\\ \Theta_2 & \text{for } \xi\to+\infty\ .\end{cases} \qquad (10)$$

As is known, a solution of Eqs. (9) and (10) exists and is unique for an appropriate temperature cutoff in the energy term [4].

For estimates we can use a simple analytical solution for the temperature profile and the flame velocity obtained by Zel'dovich and Frank-Kamenetskii (ZF) (see [4]) for the asymptotical case of infinite activation energy $\mathcal{E}\to\infty$. In terms of the dimensionless variables introduced above the temperature profile in this case becomes

$$\Theta(\xi)=\begin{cases}1+(\Theta_2-1)\exp(\xi)\ , & \xi<0\\ \Theta_2\ , & \xi>0\end{cases} \qquad (11)$$

and the corresponding eigenvalue is

$$\Lambda_{\mathrm{ZF}}=\frac{\mathcal{E}^2(\Theta_2-1)^2}{2\Theta_2^3}\exp(\mathcal{E}/\Theta_2)\ . \qquad (12)$$

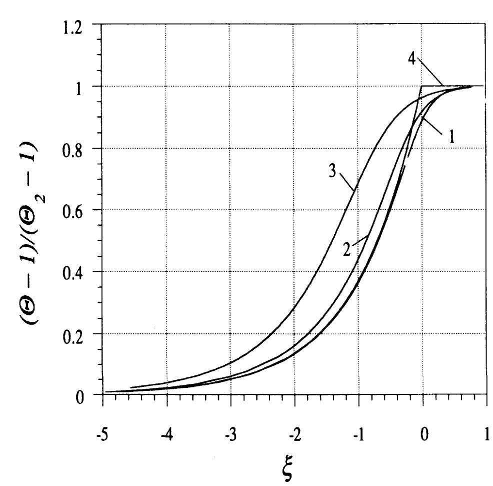

FIG. 1. Scaled profiles of the dimensionless temperature in a slow combustion wave: (1) $\mathcal{E}=70$, $\Theta_2=8$, $\ln\Lambda=14.3$; (2) $\mathcal{E}=42$, $\Theta_2=6$, $\ln\Lambda=11.7$; (3) $\mathcal{E}=21$, $\Theta_2=3$, $\ln\Lambda=10.4$. Trace (4) corresponds to the solution (11).

Figure 1 displays plots of the quantity $(\Theta-1)/(\Theta_2-1)$ for a slow-combustion wave with different values of activation energy and the thermal expansion; trace (1) is for $\mathcal{E}=70$, $\Theta_2=8$, $\ln\Lambda=14.3$; (2) $\mathcal{E}=42$, $\Theta_2=6$, $\ln\Lambda=11.7$; (3) $\mathcal{E}=21$, $\Theta_2=3$, $\ln\Lambda=10.4$. Trace (4) corresponds to the solution (11). It follows from Fig. 1 that the thickness of the flame front (the region in which thermal variation occurs) is of order unity in these dimensionless variables and varies weakly as a function of the activation energy and the thermal expansion coefficient Θ_2.

III. LINEARIZED EQUATIONS: EIGENVALUE PROBLEM

Let us consider the eigenvalue problem associated with infinitesimal perturbations about a steady planar flame. By virtue of the flow symmetry the perturbations can be taken in the form

$$\tilde{\varphi}=\tilde{\varphi}(z)\exp(\sigma t+ikx)\ , \tag{13}$$

where k is the wave number, σ is the instability growth rate, and $\tilde{\varphi}(z)$ is the amplitude of the perturbed variable. Since we are interested only in the unstable modes, in what follows we assume $\mathrm{Re}(\sigma)>0$. This implies that the perturbation amplitudes are much larger than their initial values after sufficient time has passed. Hence, we can ignore terms associated with initial transients in comparison with the asymptotic behavior of the eigenfunctions.

For the subsonic flow, the perturbed density and temperature to within terms of order M^2 are related by

$$\tilde{\rho}/\rho=-\tilde{T}/T\ . \tag{14}$$

It can also be shown (see Appendix A) that when the condition Le $=1$ holds, the perturbations $\tilde{a}$ in the fuel concentration are related to the temperature perturbations through

$$\tilde{a}=-\frac{c_p}{Q}\tilde{T}\ . \tag{15}$$

When we use Eqs. (14) and (15) and relation (8), the linearized equations (2)–(4) for the perturbations take the form

$$\frac{d\tilde{j}}{d\xi}=KS\frac{\tilde{\Theta}}{\Theta^2}-K\frac{\tilde{v}}{\Theta}\ , \tag{16}$$

$$\frac{d\tilde{v}}{d\xi}=-KS\frac{\tilde{v}}{\Theta}+K\tilde{\mathcal{P}}-2K\Theta\tilde{j}-K\tilde{\Theta}\ , \tag{17}$$

$$\frac{d\tilde{\mathcal{P}}}{d\xi}=-KS\tilde{j}-K\tilde{v}\ , \tag{18}$$

$$\frac{d^2\tilde{\Theta}}{d\xi^2}-\frac{d\tilde{\Theta}}{d\xi}-\Lambda\frac{\tilde{\Theta}}{\Theta^2}\left[\Theta_2-\frac{\mathcal{E}}{\Theta}(\Theta_2-\Theta)\right]\exp(-\mathcal{E}/\Theta)=\tilde{j}\frac{d\Theta}{d\xi}+KS\frac{\tilde{\Theta}}{\Theta}+K^2\tilde{\Theta}\ , \tag{19}$$

where the following definitions have been introduced for the dimensionless perturbations of the temperature $\tilde{\Theta}$, the transverse velocity $\tilde{v}$, mass flow $\tilde{j}$, and the dynamic pressure $\tilde{\mathcal{P}}$:

$$\tilde{\Theta}=\frac{\tilde{T}}{T_1}\ ,\quad \tilde{v}=i\frac{\tilde{u}_x}{u_{z1}}\ ,\qquad \tilde{j}=\frac{\tilde{\rho}u_z+\rho\tilde{u}_z}{\rho_1 u_{z1}}\ ,\quad \tilde{\mathcal{P}}=\frac{\tilde{P}+\tilde{\rho}u_z^2+2\rho u_z\tilde{u}_z}{\rho_1 u_{z1}^2}\ , \tag{20}$$

along with dimensionless quantities

$$K=k\Delta\ ,\quad S=\frac{\sigma}{ku_{z1}} \tag{21}$$

for the wave number and the growth rate, respectively.

The boundary conditions satisfied by Eqs. (16)–(19) require that all the perturbed quantities vanish for $\xi\to\pm\infty$. The boundary conditions can be imposed at finite displacement $\xi=\xi_1$ and $\xi=\xi_2$, which are far enough from the combustion zone that at ξ_1 and ξ_2 the flows of the fuel mixture and the combustion products, respectively, can be regarded as uniform. The conditions under which the unperturbed flow can be assumed to be uniform, i.e., $\Theta=\Theta_1=1$ for $\xi\le\xi_1$, and $\Theta=\Theta_2$ for $\xi\ge\xi_2$, are determined by the inequality

$$\left|\frac{d(\ln\Theta)}{d\xi}\right|\ll\min\{1,K\}\quad\text{for } \xi=\xi_1,\xi_2\ . \tag{22}$$

In the regions of uniform flow for $\xi<\xi_1$ and $\xi>\xi_2$ the solutions of the linearized equations can be expressed as a superposition of exponentials of the form

$$\tilde{\varphi}(\xi)=\tilde{\varphi}\exp(\mu\xi)\ , \tag{23}$$

where the argument of the exponential in (23) is chosen to satisfy $\mu>0$ for $\xi<\xi_1$, and $\mu<0$ for $\xi>\xi_2$.

Substituting the perturbations in the form (23) in Eqs. (16)–(19), we obtain for the uniform flows

$$(\mu^2-K^2)(KS+\mu\Theta)\tilde{j}=K(KS/\Theta+\mu)(\mu S/\Theta+K)\tilde{\Theta}\ , \tag{24}$$

$$\left\{\mu^2-\mu-\frac{\Lambda}{\Theta^2}\left[\Theta_2-\frac{\mathcal{E}}{\Theta}(\Theta_2-\Theta)\right]\times\exp(-\mathcal{E}/\Theta)-K^2-KS/\Theta\right\}\tilde{\Theta}=0\ . \tag{25}$$

From the fact that the solutions must vanish in the uniform regions at $\xi\to\pm\infty$, we find the following solutions of Eqs. (24)–(25).

Ahead of the combustion zone, for $\xi<\xi_1$: acoustic mode, $\tilde{\varphi}=\tilde{\varphi}_s$, $\mu=K>0$,

$$\tilde{j}_s(\xi)=\tilde{j}_s\exp(K\xi)\ ,\quad \tilde{v}_s(\xi)=-\tilde{j}_s(\xi)\ ,\qquad \tilde{\mathcal{P}}_s(\xi)=-(S-1)\tilde{j}_s(\xi)\ ,\quad \tilde{\Theta}_s(\xi)=0\ ; \tag{26}$$

thermal mode, $\tilde{\varphi}=\tilde{\varphi}_T$, $\mu=\mu_T=\frac{1}{2}+\sqrt{\frac{1}{4}+SK+K^2}>0$,

$$\tilde{\Theta}_T(\xi)=\tilde{\Theta}_T\exp(\mu_T\xi)\ ,\quad \tilde{\mathcal{P}}_T(\xi)=K^2\frac{1-S^2}{\mu_T^2-K^2}\tilde{\Theta}_T(\xi)\ ,\qquad \tilde{j}_T(\xi)=K\frac{K+\mu_T S}{\mu_T^2-K^2}\tilde{\Theta}_T(\xi)\ ,\quad \tilde{v}_T(\xi)=-\frac{K\mu_T+KS}{\mu_T^2-K^2}\tilde{\Theta}_T(\xi)\ . \tag{27}$$

Here we have used the fact that for $\xi\to-\infty$ we have

$\Theta_1 \equiv 1$ and $\exp(-\mathcal{E}) \approx 0$.

Behind the combustion zone, for $\xi > \xi_2$: acoustic mode, $\tilde{\varphi}=\tilde{\varphi}_a$, $\mu=-K$,

$$\tilde{j}_a(\xi)=\tilde{j}_a \exp(-K\xi) , \quad \tilde{v}_a(\xi)=\Theta_2 \tilde{j}_a(\xi) , \quad \tilde{P}_a(\xi)=(S+\Theta_2)\tilde{j}_a(\xi) , \quad \tilde{\Theta}_a(\xi)=0 ; \tag{28}$$

vorticity mode: $\tilde{\varphi}=\tilde{\varphi}_v$, $\mu=-KS/\Theta_2$,

$$\tilde{j}_v(\xi)=\tilde{j}_v \exp(-KS\xi/\Theta_2) , \quad \tilde{v}_v(\xi)=S\tilde{j}_v(\xi) , \quad \tilde{P}_v(\xi)=2\Theta_2\tilde{j}_v(\xi) , \quad \tilde{\Theta}_v(\xi)=0 ; \tag{29}$$

thermochemical mode: $\tilde{\varphi}=\tilde{\varphi}_c$,

$$\mu=\mu_c=\frac{1}{2}-\left[\frac{1}{4}+K^2+\frac{KS}{\Theta_2}+\frac{\Lambda}{\Theta_2}\exp(-\mathcal{E}/\Theta_2)\right]^{1/2}<0 ,$$

$$\tilde{\Theta}_c(\xi)=\tilde{\Theta}_c \exp(\mu_c\xi) , \quad \tilde{v}_c(\xi)=-\frac{K}{\Theta_2}\frac{KS+\mu_c\Theta_2}{\mu_c^2-K^2}\tilde{\Theta}_c(\xi) ,$$

$$\tilde{j}_c(\xi)=\frac{K}{\Theta_2^2}\frac{K\Theta_2+\mu_c S}{\mu_c^2-K^2}\tilde{\Theta}_c(\xi) , \tag{30}$$

$$\tilde{P}_c(\xi)=\frac{K^2}{\Theta_2^2}\frac{\Theta_2^2-S^2}{\mu_c^2-K^2}\tilde{\Theta}_c(\xi) .$$

The boundary conditions for Eqs. (16)–(19) at $\xi=\xi_1$ and $\xi=\xi_2$ are that the solution at the positions ξ_1 and ξ_2 is represented by a superposition of the eigenfunctions (26), (27), and (28)–(30), respectively,

$$\tilde{\varphi}(\xi_1)=\tilde{\varphi}_s+\tilde{\varphi}_T , \tag{31}$$

$$\tilde{\varphi}(\xi_2)=\tilde{\varphi}_a+\tilde{\varphi}_v+\tilde{\varphi}_c . \tag{32}$$

Equations (31) and (32) constitute three algebraic equations for the components of the functions $\tilde{\varphi}(\xi_1)$ and two equations for the components of the functions $\tilde{\varphi}(\xi_2)$. The system of Eqs. (16)–(19) together with the boundary conditions (31) and (32) completely specify the eigenvalue problem for the stability of a steady planar combustion wave against perturbations of the form (13).

IV. THE FLAME FRONT AS A SURFACE OF DISCONTINUITY: INSTABILITY AGAINST LONG-WAVELENGTH PERTURBATIONS

Let us consider the perturbations of the wavelength much larger than the flame thickness $K \ll 1$. In this section we shall show that the Darrieus-Landau supplementary condition for the discontinuity model is the rigorous consequence of the complete spectral problem (16)–(19), (31), and (32).

For perturbations of long wavelengths the different modes (26)–(30) vary on quite different length scales. The characteristic length scales of the hydrodynamic modes $\tilde{\varphi}_s(\xi)$, $\tilde{\varphi}_v(\xi)$, $\tilde{\varphi}_a(\xi)$ are of order K^{-1}, while for the thermochemical modes $\tilde{\varphi}_T(\xi)$ and $\tilde{\varphi}_c(\xi)$ these scales are comparable with the size of the regions of thermal conduction and reaction, i.e., smaller than or of order unity in dimensionless variables. Because the thermal modes decay rapidly as a function of distance from the combustion zone, the boundary conditions imposed on the perturbed hydrodynamic variables $\tilde{j}$, $\tilde{v}$, $\tilde{P}$ depend only on the structure of the hydrodynamic modes:

$$\tilde{v}_1=-\tilde{j}_1 , \quad \tilde{P}_1=-(S-1)\tilde{j}_1 \quad (\text{for } \xi=\xi_1) , \tag{33}$$

$$\tilde{v}_2+\tilde{P}_2-(S+2\Theta_2)\tilde{j}_2=0 \quad (\text{for } \xi=\xi_2) . \tag{34}$$

The boundary conditions for the perturbed temperature $\tilde{\Theta}$ and its derivative $d\tilde{\Theta}/d\xi$ are defined by the thermochemical modes; they are required to vanish exponentially over the energy release and thermal conductivity regions.

Let us consider the solution of Eqs. (16)–(19) with the boundary conditions (33)–(34) in the long-wavelength approximation. We write down Eq. (19) for the perturbed temperature in the form

$$\hat{F}[\tilde{\Theta}]=\tilde{j}\frac{d\Theta}{d\xi}+KS\frac{\tilde{\Theta}}{\Theta}+K^2\tilde{\Theta} , \tag{35}$$

where

$$\hat{F}\equiv\frac{d^2}{d\xi^2}-\frac{d}{d\xi}-\frac{\Lambda}{\Theta^2}[\Theta_2-\mathcal{E}(\Theta_2-\Theta)]\exp(-\mathcal{E}/\Theta) . \tag{36}$$

Note that

$$\hat{F}\left[\frac{d\Theta}{d\xi}\right]=0 . \tag{37}$$

It can be shown (see Appendix B) that the first and second terms on the right-hand side of Eq. (35) are of the same order, i.e., in the limit $K \to 0$ the solution of (35) to zeroth order of K is

$$\tilde{\Theta}=\zeta_T\frac{d\Theta}{d\xi} , \tag{38}$$

which obviously satisfies the boundary conditions imposed on $\tilde{\Theta}$ and $d\tilde{\Theta}/d\xi$. Expression (38) means that the flame is shifted in the ξ direction by an amount $\delta\xi=\zeta_T$. Using formula (38) for the perturbed temperature, we obtain the integrals of Eqs. (16)–(18) to within terms $O(K)$:

$$\tilde{j}-\tilde{j}_1=KS\zeta_T\frac{\Theta-1}{\Theta} , \tag{39}$$

$$\tilde{v}-\tilde{v}_1=-K\zeta_T(\Theta-1) , \tag{40}$$

$$\tilde{P}=\tilde{P}_1 . \tag{41}$$

Substituting $\tilde{j}$ from Eq. (39) on the right-hand side of Eq. (35) and taking into account the terms which are first order in K, we find

$$\hat{F}[\tilde{\Theta}]=(KS+\tilde{j}_1/\zeta_T)\tilde{\Theta} . \tag{42}$$

The function $\tilde{\Theta}=\zeta_T d\Theta/d\xi$ is the eigenfunction of (42) for the eigenvalue $KS+\tilde{j}_1/\zeta_T=0$; it is the unique eigenvalue for which instability can develop ($\text{Re}\sigma>0$).

Thus we obtain the condition which relates the amplitudes of the perturbations $\tilde{j}_1$ and ζ_T,

$$KS+\tilde{j}_1/\zeta_T=0 . \tag{43}$$

The consistency conditions (39)–(41), related to the downstream $\xi=\xi_2$, together with (33)–(34) and (43), yield the dispersion relation (1) for the perturbation growth rate

$$S_0=\frac{\sigma}{ku_{z1}}=\frac{\Theta_2}{\Theta_2+1}\left[\left[\Theta_2+1-\frac{1}{\Theta_2}\right]^{1/2}-1\right] . \quad (44)$$

The physical content of this long-wavelength approximation becomes evident if we write down relations (43) and (39)–(41), related to the downstream, in dimensional variables,

$$\tilde{\rho}_1 u_{z1}+\rho_1\tilde{u}_{z1}-\rho_1\sigma\zeta=0 , \quad (45)$$

$$[\tilde{\rho}u_z+\rho\tilde{u}_z]-\sigma\zeta[\rho]=0 , \quad (46)$$

$$[\tilde{u}_x+ik\zeta u_z]=0 , \quad (47)$$

$$[\tilde{P}+\tilde{\rho}u_z^2+2\rho u_z\tilde{u}_z]=0 , \quad (48)$$

where $[\psi]\equiv\psi_2-\psi_1$, represents the change in a quantity across the layer $[\xi_2,\xi_1]$ and we have introduced $\zeta=-\zeta_T\Delta$.

It is clear that conditions (46)–(48) represent the matching conditions which follow from the conservation laws if we regard the transition layer $[\xi_2,\xi_1]$ as a layer of zeroth thickness, i.e., a surface of discontinuity. Hence, ζ is nothing else but the small displacement of the surface of discontinuity in the z direction, and Eq. (45) is the Darrieus-Landau condition that the normal velocity of the flame front is unchanged [1]. Indeed for an incompressible gas $\tilde{\rho}_1=0$ and Eq. (45) goes over to

$$\tilde{u}_{z1}=\partial\zeta/\partial t=0 . \quad (49)$$

V. THE SUPPRESSION OF THE FLAME INSTABILITY

The expression (44) is the first-order term in the power series expansion in K of the perturbation growth rate. In this section we obtain the expression for the growth rate within the second-order terms in K.

When we take into account relation (43), we see that the function (38) is a solution of Eq. (35) to within terms of order K^2. Writing down the integrals of Eqs. (16)–(18) to within terms $\sim K^2$ we have

$$\tilde{j}=\tilde{j}_1+KS\zeta_T\frac{\Theta-1}{\Theta}-K\int_{\xi_1}^{\xi}\frac{\tilde{v}_0}{\Theta}d\eta , \quad (50)$$

$$\tilde{v}=-\tilde{j}_1-K\zeta_T(\Theta-1)-KS\int_{\xi_1}^{\xi}\frac{\tilde{v}_0}{\Theta}d\eta+K\int_{\xi_1}^{\xi}\tilde{\mathcal{P}}_0 d\eta$$
$$-2K\int_{\xi_1}^{\xi}\Theta\tilde{j}_0 d\eta , \quad (51)$$

$$\tilde{\mathcal{P}}=-(S-1)\tilde{j}_1-KS\int_{\xi_1}^{\xi}\tilde{j}_0 d\eta-K\int_{\xi_1}^{\xi}\tilde{v}_0 d\eta , \quad (52)$$

where $\tilde{j}_0$, $\tilde{v}_0$, and $\tilde{\mathcal{P}}_0$ are the zeroth-order terms in the expansion of $\tilde{j}$, $\tilde{v}$, and $\tilde{\mathcal{P}}$ in powers of K, defined by (39)–(41), including the boundary conditions (33). Substituting $\tilde{j}$ from (50) on the right-hand side of (35) we find

$$\hat{F}(\tilde{\Theta})=[(1-K\xi_1)\tilde{j}_1/\zeta_T+SK]\tilde{\Theta}+K^2 f(\xi)\tilde{\Theta} , \quad (53)$$

where

$$f(\xi)\equiv 1-S\frac{\xi}{\Theta}+\frac{\Theta-1}{\Theta}\xi-(S+1)J(\xi) ,$$

$$J(\xi)=\int_{\xi_1}^{\xi}\frac{\eta}{\Theta^2}\frac{d\Theta}{d\eta}d\eta . \quad (54)$$

Equation (53) can be reduced to an equation with a Hermitian operator by means of the substitution $\tilde{\Theta}=\tilde{\psi}\exp(\xi/2)$ (see Ref. [4]).

If we treat the term $K^2 f(\xi)\tilde{\Theta}$ as a small correction and use the standard techniques of perturbation theory employed in quantum mechanics [22], we find to within terms of order K^2

$$(1-K\xi_1)\tilde{j}_1+[SK+K^2\langle f\rangle]\zeta_T=0 , \quad (55)$$

where

$$\langle f\rangle=\frac{\int_{\xi_1}^{\xi_2}f(\xi)(d\Theta/d\xi)^2\exp(-\xi)d\xi}{\int_{\xi_1}^{\xi_2}(d\Theta/d\xi)^2\exp(-\xi)d\xi} . \quad (56)$$

Writing (50)–(52) at the point $\xi=\xi_2$ and taking into account (33), (34), and (55), we find

$$S=S_0(1-K/K_c) , \quad (57)$$

where S_0 is defined by Eq. (44) and

$$K_c^{-1}=\frac{1}{2S_0\sqrt{\Theta_2+1-1/\Theta_2}}\left\{2(S_0+\Theta_2)\left[1+\langle\xi\rangle-(1+S_0)\left\langle\frac{\xi}{\Theta}+J(\xi)\right\rangle\right]+(S_0^2+2S_0\Theta_2+2\Theta_2)J(\xi_2)-\int_{\xi_1}^{\xi_2}\eta\frac{d\Theta}{d\eta}d\eta\right\} . \quad (58)$$

Since in the points ξ_1,ξ_2 the flow can be treated as uniform, we can replace the coordinates by the infinite coordinates $-\infty$ and $+\infty$ in all integrals of (58). It is worth emphasizing that the results (57) and (58) are independent of the particular form of energy release (5), that's why they are valid for any activation energy.

For the case of infinite activation energy using expression (11) we obtain

$$K_c^{-1}=\frac{(\Theta_2-1)}{2S_0\sqrt{\Theta_2+1-1/\Theta_2}}\times\left\{1+\frac{S_0^2(\Theta_2+1)+4S_0\Theta_2+2\Theta_2}{(\Theta_2-1)^2}\ln(\Theta_2)\right\} . \quad (59)$$

In this particular case the expression (59) coincides with the results obtained in Refs. [15–18].

The value of K_c slightly depends upon the thermal expansion factor Θ_2 and the approximate value is $K_c \approx 0.3$, for example, from (59) $K_c = 0.31$ for $\Theta_2 = 6$; and $K_c = 0.28$ for $\Theta_2 = 4$. Thus, the length scale of the fastest perturbations is $\lambda_f = 4\pi\Delta/K_c \approx 40\Delta$, which is about two orders of magnitude greater than the flame thickness.

VI. NUMERICAL SOLUTION OF THE EIGENVALUE PROBLEM

For general situations the stability problem for the flame with complicated chemical and transport processes can be solved only numerically. In this section we obtain the numerical solution using the method which can be easily generalized for a more sophisticated system of equations. The numerical solution of Eqs. (16)–(19) for the eigenvalues with the boundary conditions (31) and (32) was found by iteration. For a given approximate value $S = S_n$ the system of differential equations (16)–(19) was integrated twice over the interval $[\xi_1, \xi^*]$ with boundary conditions $\tilde{\varphi}_1(\xi_1) = \tilde{\varphi}_s$ and $\tilde{\varphi}_2(\xi_2) = \tilde{\varphi}_T$, and three times on the interval $[\xi^*, \xi_2]$ with boundary conditions $\tilde{\varphi}_3(\xi_2) = \tilde{\varphi}_a$, $\tilde{\varphi}_4(\xi_2) = \tilde{\varphi}_v$, $\tilde{\varphi}_5(\xi_2) = \tilde{\varphi}_c$. Then, a new approximate value $S = S_{n+1}$ was found by Newton's method,

$$S_{n+1} = S_n - D(S_n, \xi^*)/(\partial D/\partial S) , \tag{60}$$

where

$$D(S,\xi) = \det[\tilde{\varphi}_1(\xi,S), \tilde{\varphi}_2(\xi,S), \tilde{\varphi}_3(\xi,S), \tilde{\varphi}_4(\xi,S), \varphi_5(\xi,S)] , \tag{61}$$

and the derivative in the denominator is approximated by a finite difference.

This method of calculating the eigenvalues is based on the fact that the desired eigenfunctions must be expandable at $\xi = \xi_1$ in the modes $\tilde{\varphi}_s$ and $\tilde{\varphi}_T$, and at $\xi = \xi_2$ in terms of the modes $\tilde{\varphi}_a$, $\tilde{\varphi}_v$, $\tilde{\varphi}_c$, respectively. In other words, the solution represented as the superposition of the functions $\tilde{\varphi}_s$ and $\tilde{\varphi}_T$ at $\xi = \xi_1$ must go over to a solution that can be represented as a superposition of the functions $\tilde{\varphi}_a$, $\tilde{\varphi}_v$, $\tilde{\varphi}_c$ at $\xi = \xi_2$. This requirement reduces to the condition that the determinant (61) vanish at any point in the interval $[\xi_1, \xi_2]$.

As the point ξ^* where the solutions from the left and the right are "matched" we used the point corresponding to the maximum heat production rate in the thermal conduction equation. This choice of the matching point ensures the greatest accuracy in finding the eigenvalue, since in both cases the integration from the points $\xi = \xi_1$ and $\xi = \xi_2$ is along the attractive trajectories.

Eigenvalues of Eqs. (16)–(19) can also be sought in the complex plane. The calculations show that this problem has no complex eigenvalues.

The results of the numerical solution of the problem are shown in Figs. 2 and 3. Figure 2 shows the calculated instability growth rates, scaled by the corresponding value given by Eq. (1), as a function of the perturbation

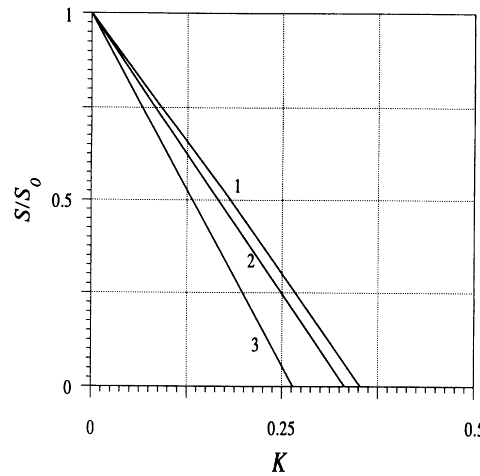

FIG. 2. Scaled growth rate $\sigma/(ku_1 S_0) = S/S_0$ for the flame instability as a function of the dimensionless wave numbers for the unperturbed flows in the flame shown in Fig. 1. The numbers labeling the traces in the figure correspond to the temperature profiles in Fig. 1.

wave number. This figure shows clearly that the instability growth rates are in a good agreement with the Darrieus-Landau solution (1), independently of the activation energy and the thermal expansion factor. It should be noted, however, that a marked deviation in the growth rate from the value given by Eq. (1) occurs even for $\lambda > 100\Delta$, while for $\lambda \approx 20\Delta$ the growth rate vanishes. The calculations reveal that, to within the accuracy of the numerical method, the growth rates depend weakly on the activation energy and the thermal expansion factor. Figure 3 shows the dimensionless growth rate $\sigma\Delta/u_1$ as a function of the wave number K for the different expansion factors and activation energies corresponding to Fig. 1. The results of the numerical calculations shown in Fig. 3 demonstrate good agreement with the analytical solution Eq. (58).

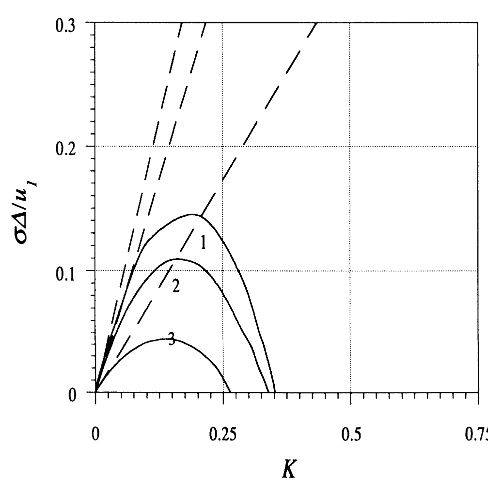

FIG. 3. Dimensionless growth rate as a function of wave number for the unperturbed flows shown in Fig. 1.

VII. CONCLUSION

The complete rigorous solution is obtained for the problem of the flame stability, including the chemical kinetic of the reaction and the thermal conduction of the gas. It is shown that in the limit of long-wavelength perturbations ($k \to 0$) the condition that the flame velocity relative to the unperturbed gas remains unchanged follows from the complete system of linearized equations. Solution of the stability problem is obtained for the case Le=1 and for arbitrary activation energy in explicit analytical form and numerically. An obtained analytical solution of the problem does not depend on the particular function of the energy release and can be easily generalized for the wide class of similar problems. For the infinite activation energy the obtained solution is in agreement with the results of the papers [15–18] for the case Le=1. The significant reduction in the instability growth rate relative to the values predicted by Eq. (1), even for perturbations with wavelength much greater than the thickness of the combustion zone, causes an increase by about two orders of magnitude in estimates of the length scales of the fastest perturbations. One would also expect that for flames with the cellular structure, the scale of the cells which arise in the late nonlinear states should be comparable with the wavelength of perturbations corresponding to the maximum growth rate shown in Fig. 3.

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APPENDIX A

From the equation for the fuel concentration,

$$\rho\frac{\partial a}{\partial t}+\rho(\mathbf{u}\cdot\nabla)a=\nabla\cdot(\rho D\nabla a)-a\frac{\rho}{\tau}\exp(-E/RT) , \quad (A1)$$

and the thermal conduction equation

$$\rho\frac{\partial T}{\partial t}+\rho(\mathbf{u}\cdot\nabla)T=\nabla\cdot\left[\frac{\kappa}{c_p}\nabla T\right]+a\frac{\rho Q}{\tau c_p}\exp(-E/RT) , \quad (A2)$$

taking into account the condition $\mathrm{Le}=\rho c_p D/\kappa=1$, we have

$$\rho c_p\frac{\partial H}{\partial t}+\rho c_p(\mathbf{u}\cdot\nabla)H=\nabla\cdot(\kappa\nabla H) , \quad (A3)$$

where $H=c_pT+aQ$ is the enthalpy.

The linearized equation for the infinitesimal $\tilde{H}$ is

$$\kappa\frac{d^2\tilde{H}}{dz^2}=\rho u c_p\frac{d\tilde{H}}{dz}+(\kappa k^2+\rho c_p\sigma)\tilde{H} . \quad (A4)$$

Multiplying (A4) by $d\tilde{H}/dz$, integrating from $z=-\infty$ to $z=\infty$, and using the condition $\tilde{H}(\pm\infty)=0$, we find

$$\int_{-\infty}^{+\infty}\left[\frac{d\tilde{H}}{dz}\right]^2dz=\frac{\sigma}{2}\int_{-\infty}^{+\infty}\tilde{H}^2\frac{d\rho}{dz}dz . \quad (A5)$$

Since we have $\mathrm{Re}(\sigma)>0$ and $d\rho/dz<0$, the relation (A5) can hold only for $d\tilde{H}/dz=0$, $\tilde{H}=0$, i.e., $\tilde{a}=-(c_p/Q)\tilde{T}$, QED.

APPENDIX B

We show that $\tilde{j}(d\Theta/d\xi)$, and $SK\tilde{\Theta}$ in Eq. (35) are of the same order. Assume that

$$SK\tilde{\Theta}\ll\tilde{j}\frac{d\Theta}{d\xi} . \quad (B1)$$

Then from Eqs. (16)–(18) in the region $\xi_1<\xi<\xi_2$, to within terms of order $\kappa\tilde{j}$, we deduce the relations

$$\tilde{j}=\tilde{j}_1 , \quad (B2)$$

$$\tilde{v}=\tilde{v}_1 , \quad (B3)$$

$$\tilde{\mathcal{P}}=\tilde{\mathcal{P}}_1 . \quad (B4)$$

Substituting (B2)–(B4) in (33)–(34) we find

$$S=-\Theta_2<0 ,$$

which contradicts the assumption $S>0$.

Reversing the inequality (B1),

$$SK\frac{\tilde{\Theta}}{\Theta}\gg\tilde{j}\frac{d\Theta}{d\xi} , \quad (B5)$$

we find from (19) to within terms of order $SK\tilde{\Theta}$

$$\frac{d^2\tilde{\Theta}}{d\xi^2}-\frac{d\Theta}{d\xi}-\Lambda\Theta_2\frac{\tilde{\Theta}}{\Theta^2}\left[1+\frac{\mathcal{E}}{\Theta_2}\left[\frac{\Theta_2}{\Theta}-1\right]\right]\exp(-\mathcal{E}/\Theta)=SK\frac{\tilde{\Theta}}{\Theta} . \quad (B6)$$

This equation has no positive eigenvalues (see Ref. [4]). Thus, positive eigenvalues $S>0$ are possible only for $SK\tilde{\Theta}\sim\tilde{j}(d\Theta/d\xi)$, QED.

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